

Approximation by Polynomials with Restricted Zeros

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This paper discusses convergence properties of polynomials whose zeros lie on the real axis or in the upper half-plane. A result of Levin shows that uniform convergence of such polynomials to a non-zero limit on a complex sequence converging not too fast to a limit in the lower half-plane implies locally uniform convergence in \mathbb{C} . We give a relatively simple proof of this result and present several extensions and examples which show that the criterion in Levin's theorem is almost sharp. © 1994 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

An entire function is in $\mathcal{L}\text{-}\mathcal{P}$, the Laguerre–Pólya class, if and only if it is the local uniform limit in \mathbb{C} of a sequence of polynomials with only real zeros. An entire function is in $\mathcal{P}\text{-}\mathcal{O}$, the Pólya–Obrechhoff class, if and only if it is the local uniform limit in \mathbb{C} of a sequence of polynomials whose zeros lie in the closed upper half-plane. Each of $\mathcal{L}\text{-}\mathcal{P}$ and $\mathcal{P}\text{-}\mathcal{O}$ consists of functions of order at most 2 and of a particular form, see [2].

Polynomials with only real zeros, or with only zeros in the closed upper half-plane, have the interesting property that uniform convergence of a sequence of such polynomials on a suitable small set implies locally uniform convergence in \mathbb{C} . The first result in the former case was obtained by Pólya [6] who showed that a domain in \mathbb{C} is suitable. The corresponding result in the latter case was proved by Lindwart and Pólya [4]. They proved the following:

THEOREM 1.1. (Lindwart–Pólya). *Let (p_n) be a sequence of polynomials all of whose zeros are in the closed upper half-plane. Suppose that (p_n) converges uniformly on a domain in \mathbb{C} which meets the lower half-plane, to a*

limit function not identically zero. Then (p_n) converges locally uniformly to a function in $\mathcal{P}-\mathcal{O}$.

Later Bernstein [1], Korevaar and Loewner [2], and Levin [3] gave various extensions of these results. The most general result was given by Levin, which we state in the following form:

THEOREM 1.2. (Levin). *Let (p_n) be a sequence of polynomials whose zeros are in the closed upper half-plane. Let (z_j) be a complex sequence converging to ζ with $\text{Im } \zeta \leq 0$, such that for some fixed number $r \in (0, 1)$ we have*

$$r|z_j - \zeta| \leq |z_{j+1} - \zeta| \quad (j \geq 1). \quad (1)$$

Suppose that (p_n) converges uniformly on $K = \{z_j\} \cup \{\zeta\}$ to a limit function which is non-zero at ζ . Then (p_n) converges locally uniformly in \mathbf{C} to a function in $\mathcal{P}-\mathcal{O}$.

The results of Bernstein and Levin were overlooked by many researchers. Clearly, Korevaar and Loewner were not aware of Bernstein's paper. The present authors were not aware of the work of Bernstein and Levin for some time, and an earlier version of this paper contained a result that was in essence Levin's Theorem 1.2 for the case of polynomials with real zeros.

Before we came across Levin's paper [3] we were preparing a second paper in which we proved independently, among other things, the full result of Levin. We give this proof below. It is of interest since it is more concise (and perhaps more elegant) than that of Levin who, unlike us, uses estimates on the coefficients of the polynomials.

From Theorem 1.2 we deduce quite easily an interesting corollary.

COROLLARY 1.3. *Let (p_n) be a sequence of polynomials whose zeros are in the closed upper half-plane and suppose that (p_n) converges to a function f in measure (Lebesgue) on some set $E \subset \mathbf{R}$ of positive measure with $\{x \in E | f(x) \neq 0\}$ of positive measure. Then f is equal almost everywhere on E to a function in $\mathcal{P}-\mathcal{O}$ and (p_n) converges locally uniformly in \mathbf{C} .*

Of course, Theorem 1.2 also applies to polynomials having only real zeros. In this case it is perhaps natural to deal with real sequences (z_j) . Our next result shows that in this special case the result of Theorem 1.2 is not far from being the best possible.

THEOREM 1.4. *Let (x_j) be a strictly decreasing sequence of reals such that $x_j \rightarrow \xi$ ($j \rightarrow \infty$) and*

$$\inf_j \frac{x_{j+1} - \xi}{x_j - \xi} = 0, \quad \sup_j \frac{x_{j+1} - \xi}{x_j - \xi} < 1. \quad (2)$$

Then there exists a sequence (p_n) of polynomials whose zeros are real, which converges uniformly to 1 on $K = \{x_j\} \cup \{\xi\}$, but not on any neighbourhood of ξ .

To bridge the gap between Theorems 1.2 and 1.4 we need to consider real decreasing sequences (x_j) with limit ξ satisfying

$$\inf_j \frac{x_{j+1} - \xi}{x_j - \xi} = 0, \quad \sup_j \frac{x_{j+1} - \xi}{x_j - \xi} = 1. \quad (3)$$

We have not been able to settle this case completely. We give examples of sequences (x_j) satisfying (3) for which there exist polynomials as in Theorem 1.4, and other examples for which such polynomials do not exist. For the precise conditions, see Section 4.

The next result shows that if ζ is the limit of (z_j) and $\text{Im } \zeta < 0$, then it is sufficient that the z_j approach ζ from different directions, as specified below, for the conclusion of Theorem 1.2 to hold.

THEOREM 1.5. *Let (z_j) be a complex sequence with*

$$\lim_{j \rightarrow \infty} z_j = \zeta, \quad \text{Im } \zeta < 0.$$

Suppose that there is some $r > 0$ such that for every j_0 there exist $j, k \geq j_0$ such that

$$|\sin(\phi_k - \phi_j)| \geq r,$$

where $z_j - \zeta = \rho_j e^{i\phi_j}$. Let (p_n) be a sequence of polynomials whose zeros lie in the closed upper half-plane and suppose that (p_n) converges pointwise on $K = \{z_j\} \cup \{\zeta\}$ to f , where $f(z) \neq 0$, $z \in K$. Then (p_n) converges locally uniformly in \mathbb{C} to a function in $\mathcal{P}\text{-}\mathcal{O}$.

In contrast to the proof of Theorem 1.2, the proof of Theorem 1.5 requires a more detailed look at the zeros of the polynomials p_n . It is in the same spirit as the proofs in [2].

Finally, we present some more examples to illustrate our results to supplement those discussed following the proof of Theorem 1.4.

2. LEVIN'S THEOREM

The two main ingredients which will be used in the proof of Theorem 1.2 are the Lindwärt-Pólya result stated in the Introduction, and the

classical result of Montel [5] on uniform bounded families of analytic functions, which we state for convenience as follows:

THEOREM 2.1. (Montel). *Every sequence (f_n) of analytic functions uniformly bounded on an open set U has a subsequence which converges locally uniformly in U .*

As a final remark before the proof we note that it suffices to show that a subsequence of (p_n) converges locally uniformly in \mathbf{C} , since all possible limit functions are the same as they coincide on K .

Proof of Theorem 1.2. By considering $\tilde{z}_j = z_j - \zeta$, $\tilde{f}(z) = f(z + \zeta)/f(\zeta)$, $\tilde{p}_n(z) = p_n(z + \zeta)/p_n(\zeta)$, we may assume

$$\zeta = 0, \quad f(0) = 1, \quad p_n(0) = 1 \quad (n \geq 1).$$

For every n , let $a_n > 0$ be such that

$$\max_{|z|=a_n} |p_n(z)| = 2. \quad (4)$$

[Since $p_n(0) = 1$ and assuming p_n is not a constant, the number a_n is uniquely determined. If p_n is a constant then $a_n = \infty$.]

If the sequence (a_n) remains bounded away from 0, then there exists a neighbourhood U of 0 such that the polynomials p_n are uniformly bounded on U . Theorem 2.1 shows that some subsequence of (p_n) converges locally uniformly on U . Hence, passing to that subsequence, we may assume that (p_n) converges locally uniformly in U . The limit function does not vanish identically, since $p_n(0) = 1$ for every n . Now Theorem 1.1 implies that the subsequence converges locally uniformly in \mathbf{C} . So the theorem follows provided the sequence (a_n) is bounded away from 0.

To obtain a contradiction suppose (a_n) is not bounded away from 0. Passing to a subsequence we thus assume

$$\lim_{n \rightarrow \infty} a_n = 0. \quad (5)$$

We introduce auxiliary polynomials q_n given by

$$q_n(z) = p_n(a_n z). \quad (6)$$

The polynomials q_n have their zeros in the closed upper half-plane and are uniformly bounded by the value 2 on the unit disk. Using the same arguments as above, we find a subsequence of (q_n) which converges locally uniformly in \mathbf{C} . Passing to this subsequence we find that (q_n) converges locally uniformly in \mathbf{C} . We denote the limit function by q .

Next, let for $k \in \mathbb{N}$, A_k be the annular region given by

$$A_k = \{z \in \mathbb{C} | r^{k+1} \leq |z| \leq r^k\}.$$

Fix k . Because of (1) and (5) there exists, for all n sufficiently large [in fact, all n such that $a_n \leq |z_1|$], an index j_n such that

$$a_n r^{k+1} \leq |z_{j_n}| \leq a_n r^k, \tag{7}$$

that is, $a_n^{-1} z_{j_n} \in A_k$. By (5) and (7) we have $\lim_{n \rightarrow \infty} z_{j_n} = 0$. Then

$$|q_n(a_n^{-1} z_{j_n}) - 1| = |p_n(z_{j_n}) - 1| \leq |p_n(z_{j_n}) - f(z_{j_n})| + |f(z_{j_n}) - f(0)|,$$

which tends to 0 as $n \rightarrow \infty$, since (p_n) converges uniformly on K to f and f is continuous at 0. It follows that q assumes the value 1 in the region A_k . Since k is arbitrary, $q(z) = 1$ at an infinite number of points with 0 as limit point. Since q is an entire function, q has to be identically 1.

However, from (4) and (6) it follows that $|q_n(z)| = 2$ for some z on the unit circle $|z| = 1$ and therefore $|q(z)| = 2$ somewhere on the unit circle. This is a contradiction. Hence (5) cannot hold and the theorem is proved. ■

Proof of Corollary 1.3. There is a set $E_0 \subset E$ of positive measure on which $f(x)$ is everywhere non-zero and on which $q_n \rightarrow f$ ($n \rightarrow \infty$) uniformly for some subsequence (q_n) of (p_n) , see [8, p. 92], so that f is continuous on E_0 . Almost all points of E_0 are points of density, see [9, p. 371], and choose such a point $a \in E_0$ so that

$$\lim_{h \rightarrow 0^+} \frac{\text{meas}(E_0 \cap [a - h, a + h])}{2h} = 1.$$

Choose $h_0 > 0$ so that for $0 < h \leq h_0$,

$$\frac{\text{meas}(E_0 \cap [a - h, a + h])}{2h} > \frac{7}{8}.$$

Define $h_j = 2^{-j} h_0$ and consider $E_0 \cap [a + h_j/2, a + 3h_j/4]$. If this set is empty, then

$$\text{meas}(E_0 \cap [a - h_j, a + h_j]) \leq 2h_j - \frac{1}{4}h_j = \frac{7}{8}h_j$$

and this is impossible by the choice of h_j and h_0 . Hence there is a point of E_0 in $[a + h_j/2, a + 3h_j/4]$ and we choose such a point and denote it by x_j .

The sequence of points (x_j) satisfies the inequalities

$$h_j/2 \leq x_j - a \leq 3h_j/4, \quad j \geq 0.$$

These inequalities ensure that (x_j) is a strictly decreasing sequence with limit a and that

$$\frac{x_{j+1} - a}{x_j - a} \geq \frac{h_{j+1}/2}{h_j} = \frac{1}{4}.$$

Hence (x_j) satisfies the condition (1) of Theorem 1.2 with $r = 1/4$. By Theorem 1.2, (q_n) converges locally uniformly in \mathbf{C} to a function in $\mathcal{L}\text{-}\mathcal{P}$ that agrees with f a.e. on E .

By the same argument every subsequence of (p_n) contains a subsequence that converges locally uniformly in \mathbf{C} to the same function in $\mathcal{L}\text{-}\mathcal{P}$. It follows that (p_n) converges locally uniformly in \mathbf{C} and the corollary is proved. ■

Remark. A similar argument proves the corresponding result for pointwise convergence.

3. THE CASE $\liminf((x_{j+1} - \xi)/(x_j - \xi)) = 0$, $\limsup((x_{j+1} - \xi)/(x_j - \xi)) < 1$

Proof of Theorem 1.4. Assume that the conditions of Theorem 1.4 are satisfied with $\xi = 0$. Let $\varepsilon_j = x_{j+1}/x_j$. Choose a subsequence (j_n) of the positive integers for which $\varepsilon_{j_n} \rightarrow 0$ ($n \rightarrow \infty$).

Consider the polynomials

$$p_n(z) = 1 + C \frac{z}{x_{j_n}} \prod_{k=1}^{j_n} \left(1 - \frac{z}{x_k}\right), \quad (8)$$

where $C > 0$ will be specified later. For $1 \leq j \leq j_n$, $p_n(x_j) = 1$; and for $j > j_n$,

$$|p_n(x_j) - 1| = C \frac{x_j}{x_{j_n}} \prod_{k=1}^{j_n} \left(1 - \frac{x_j}{x_k}\right) \leq C \varepsilon_{j_n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore $p_n \rightarrow 1$ ($n \rightarrow \infty$) uniformly on K . Since $p'_n(0) = C/x_{j_n} \rightarrow \infty$ ($n \rightarrow \infty$) the polynomials do not converge uniformly in any neighbourhood of 0.

We now show that a suitable choice of C ensures that each p_n has only real zeros by proving that the minima of $p_n(x) - 1$, $x \in \mathbf{R}$, are less than -1 , provided that C is large enough. This is achieved by showing that for $k \leq j_n$, $k \geq 1$, each interval $[x_k, x_{k-1}]$ contains a point ξ_k such that

$$|p_n(\xi_k) - 1| \geq 1. \tag{9}$$

Let $s > 1$ be such that $\varepsilon_j < 1/s$ for every j . We set $\xi_k = sx_k$ for $1 \leq k \leq j_n$. For $j < k$ we have that

$$\begin{aligned} \left| 1 - \frac{\xi_k}{x_j} \right| &= 1 - s \frac{x_k}{x_j} = 1 - s \frac{x_k}{x_{k-1}} \frac{x_{k-1}}{x_{k-2}} \dots \frac{x_{j+1}}{x_j} \\ &= 1 - s\varepsilon_{k-1} \dots \varepsilon_j \geq 1 - s^{j+1-k}; \end{aligned} \tag{10}$$

and for $j \geq k$ we have that

$$\begin{aligned} \left| 1 - \frac{\xi_k}{x_j} \right| &= s \frac{x_k}{x_j} - 1 = s \frac{x_k}{x_{k+1}} \cdot \frac{x_{k+1}}{x_{k+2}} \dots \frac{x_{j-1}}{x_j} - 1 \\ &= s(\varepsilon_k \dots \varepsilon_{j-1})^{-1} - 1 \geq s^{(j-k+1)} - 1. \end{aligned} \tag{11}$$

Since $s > 1$, there is some integer k_0 so that $s^{k+1} - 1 > 1$ for $k > k_0$. Hence, from (11),

$$\prod_{j=k}^{j_n} \left| 1 - \frac{\xi_k}{x_j} \right| \geq (s - 1)^{k_0}; \tag{12}$$

and from (10),

$$\prod_{j=1}^{k-1} \left| 1 - \frac{\xi_k}{x_j} \right| \geq \prod_{j=1}^{k-1} (1 - s^{j+1-k}) \geq \prod_{\nu=0}^{\infty} (1 - s^{-\nu}) =: B > 0. \tag{13}$$

From (12) and (13) it now follows that

$$|p_n(\xi_k) - 1| = C \frac{\xi_k}{x_{j_n}} \prod_{j=1}^{j_n} \left| 1 - \frac{\xi_k}{x_j} \right| \geq CB(s - 1)^{k_0}.$$

If we now specify

$$C = \frac{2}{B(s - 1)^{k_0}}$$

then we have the conditions satisfied which ensure that each p_n has only real zeros. This proves Theorem 1.4. ■

4. THE CASE $\liminf((x_{j+1} - \xi)/(x_j - \xi)) = 0,$
 $\limsup((x_{j+1} - \xi)/(x_j - \xi)) = 1$

Theorems 1.2 and 1.4 answer to a large extent the question for what decreasing sequences (x_j) there exist polynomials having only real zeros which converge uniformly on $K = \{x_j\} \cup \{\xi\}$ to a limit function which is non-zero at ξ , but which do not converge locally uniformly in \mathbf{C} . The case

$$\liminf \frac{x_{j+1} - \xi}{x_j - \xi} = 0, \quad \limsup \frac{x_{j+1} - \xi}{x_j - \xi} = 1, \tag{14}$$

remains to be considered. We show that in this case both situations can occur.

PROPOSITION 4.1. *There exist a positive decreasing sequence (x_j) with limit ξ satisfying (14), and a sequence of polynomials (p_n) having only real zeros, such that (p_n) converges uniformly to 1 on $K = \{x_j\} \cup \{0\}$ but not locally uniformly in \mathbf{C} .*

Proof. We choose $\xi = 0$ and suppose that $x_1 > x_2 > \dots > x_{j_n} > 0$ have been found. Construct as in the proof of Theorem 1.4 the polynomial

$$p_n(z) = 1 + C_n \frac{z}{x_{j_n}} \prod_{k=1}^{j_n} \left(1 - \frac{z}{x_k}\right).$$

Then $p_n(0) = p_n(x_1) = p_n(x_2) = \dots = p_n(x_{j_n}) = 1$ and as in the proof of Theorem 1.4, for some C_n large enough the polynomial p_n has only real zeros. Then one can take $x_{j_{n+1}}$ small such that $|p_n(x) - 1| < 1/n$ for all $x \in [0, x_{j_{n+1}}]$. Then one takes $x_{j_{n+1}} > x_{j_{n+2}} > \dots > x_{j_{n+1}} > 0$ as close to one another as one likes and continues to construct p_{n+1} .

Following this procedure one arrives at a sequence (x_j) and polynomials (p_n) which satisfy the proposition. ■

For the proof of the following proposition we need a lemma.

LEMMA 4.2. *Let (p_n) be a sequence of polynomials having only real zeros. Let $a < b < c$ be real numbers such that*

- (a) every p_n is free of zeros in $[a, c]$;
- (b) $\inf_n p_n(a) > 0, \inf_n p_n(c) > 0$;
- (c) $\sup_n p_n(b) < \infty$.

Then the sequence (p_n) is normal in \mathbf{C} .

Proof. From (a) and (b) it follows that the polynomials p_n are uniformly bounded away from 0 on the interval $[a, c]$. For any polynomial p having only real zeros it is easy to show that

$$|p(z)| \geq |p(\operatorname{Re} z)|, \quad z \in \mathbf{C},$$

and so the functions $1/p_n$ are uniformly bounded in the strip $S = \{a < \operatorname{Re} z < c\}$. By Montel's Theorem 2.1, there is a subsequence $(p_{n_k}^{-1})$ which converges locally uniformly in S . The limit is identically zero or free of zeros in S , by Hurwitz' theorem. The former possibility is excluded by the condition (c). It follows that (p_{n_k}) converges locally uniformly in S to a limit function having no zeros in S . So by the result of Pólya, cf. Theorem 1.1, this subsequence is locally uniformly convergent in \mathbf{C} . ■

To construct an example of a sequence satisfying (3) for which there do not exist polynomials (p_n) having only real zeros, converging uniformly on $K = \{x_j\} \cup \{0\}$ but not locally uniformly in \mathbf{C} , we look at sequences (x_j) containing a subsequence (X_k) such that

(A) $\lim_k (X_{2k+1}/X_{2k}) = 0$ and $X_{2k} = \frac{1}{2}X_{2k-1}$.

(B) Every interval (X_{2k+1}, X_{2k}) contains no points x_j .

(C) Every interval $[X_{2k}, X_{2k-1}]$ contains $\nu_k + 2$ points of K which are equidistantly spaced, starting at X_{2k} and ending at X_{2k-1} .

(D) The number ν_k is odd, so that $(X_{2k} + X_{2k-1})/2 \in K$.

PROPOSITION 4.3. *Let $(x_j), (X_k), (\nu_k)$ be as described above. Suppose*

$$\inf_{k \rightarrow \infty} \nu_k \left(\frac{X_{2k+1}}{X_{2k}} \right)^2 > 0. \tag{15}$$

If (p_n) is a sequence of polynomials whose zeros are real and which converges uniformly on $K = \{x_j\} \cup \{0\}$ to a function which is non-zero at 0, then (p_n) is locally uniformly convergent in \mathbf{C} to a function in $\mathcal{L}\text{-}\mathcal{P}$.

Remark. The conditions of the proposition are certainly not the best possible. But it is only our aim to give an example.

Proof. We may assume $p_n(0) = 1$ for every n . If for some $\delta > 0$, p_n is free of zeros in $[0, \delta]$ for all n , then Lemma 4.2 implies that (p_n) converges locally uniformly in \mathbf{C} . Therefore we may assume that the smallest positive zero of p_n tends to 0 [passing to a subsequence if necessary].

Let x_{j_n} be the largest member of (x_j) such that p_n is free of zeros in $[0, x_{j_n}]$. Let X_{2k_n+1} be the largest X_k with odd index smaller than x_{j_n} . We have $\lim_n j_n = \infty, \lim_n k_n = \infty$.

We are going to prove that for large enough n , p_n has at least ν_{k_n} zeros in the interval (X_{2k_n}, X_{2k_n-1}) .

Let us first assume that (X_{2k_n}, X_{2k_n-1}) contains no zeros of p_n for all large enough n . Then $x_{j_n} \in (X_{2k_n+1}, X_{2k_n+1})$. The polynomials

$$Q_n^{(1)}(z) = p_n(X_{2k_n} + (X_{2k_n-1} - X_{2k_n})z)$$

are zero-free in $[0, 1]$. Since $Q_n^{(1)}(0) \rightarrow 1$, $Q_n^{(1)}(1) \rightarrow 1$, and $Q_n^{(1)}(1/2) \rightarrow 1$, cf. property (D) above, it follows by Lemma 4.2 that $(Q_n^{(1)})$ is a normal family in \mathbb{C} . It is easily seen that the only possible limit function is the constant function 1. Note however that $Q_n^{(1)}(z)$ has a zero in $[-1, 0]$ (because of property (A)). So p_n has at least one zero in (X_{2k_n}, X_{2k_n-1}) .

The interval (X_{2k_n}, X_{2k_n-1}) consists of $\nu_{k_n} + 1$ subintervals (x_j, x_{j-1}) all having length

$$\delta_n := \frac{X_{2k_n-1} - X_{2k_n}}{\nu_{k_n}}.$$

Since for large n , $p_n(x_j) > 0$ for every $x_j \in [X_{2k_n}, X_{2k_n-1}]$, p_n has an even number of zeros in every subinterval (x_j, x_{j-1}) . If $p_n(z)$ contains fewer than ν_{k_n} zeros in (X_{2k_n}, X_{2k_n-1}) we can find a subinterval $[x_{j+1}, x_{j-1}]$ free of zeros, but such that $[x_{j+1} - \delta_n, x_{j-1} + \delta_n]$ contains a zero of $p_n(z)$. Considering the polynomials

$$Q_n^{(2)}(z) = p_n(x_j + \delta_n z)$$

and arguing as above, we arrive at a contradiction. Hence there are at least ν_{k_n} zeros of $p_n(z)$ in (X_{2k_n}, X_{2k_n-1}) for all large n .

Writing

$$Q_n^{(3)}(z) = p_n(X_{2k_n+1}z)$$

we see that $Q_n^{(3)}(z)$ tends to 1 for $z = 0, 1/2, 1$ so by a now familiar argument, the sequence $(Q_n^{(3)})$ converges to 1 locally uniformly in \mathbb{C} . Hence for large n , p_n is zero-free on $[-MX_{2k_n+1}, MX_{2k_n+1}]$, where M is some fixed positive constant. We choose $M = 9$.

Now, let

$$p_n(z) = \prod_k \left(1 - \frac{z}{a_k}\right) \prod_l \left(1 + \frac{z}{b_l}\right),$$

where $a_k = a_{k,n}$ denote the positive zeros of p_n and $-b_l = -b_{l,n}$ denote the negative zeros. Put $\phi_n(x) = \log(p_n(x))$ for $|x| \leq X_{2k_n+1}$. Then it easily

follows by a Taylor expansion that $\phi_n(x) = S_n(x) - T_n(x)$ with

$$S_n(x) = \left(\sum_l \frac{1}{b_l} - \sum_k \frac{1}{a_k} \right) x,$$

$$T_n(x) = \left(A_n(x) \sum_k \frac{1}{a_k^2} + B_n(x) \sum_l \frac{1}{b_l^2} \right) \frac{x^2}{2},$$

and where for n large enough,

$$1 \leq A_n(x) \leq \left(\frac{9}{8} \right)^2, \quad \left(\frac{9}{10} \right)^2 \leq B_n(x) \leq 1 \quad (x \in [0, X_{2k_n+1}]). \quad (16)$$

Because the interval $[X_{2k_n}, X_{2k_n-1}]$ contains at least ν_{k_n} zeros, we have, from property (A),

$$\sum_k \frac{1}{a_k^2} \geq \nu_{k_n} \left(\frac{1}{X_{2k_n-1}} \right)^2 = \frac{1}{4} \nu_{k_n} \left(\frac{1}{X_{2k_n}} \right)^2.$$

By the assumption (15) this implies, again from property (A),

$$X_{2k_n+2}^2 \sum_k \frac{1}{a_k^2} = \frac{1}{4} X_{2k_n+1}^2 \sum_k \frac{1}{a_k^2} \geq \frac{1}{16} \nu_{k_n} \left(\frac{X_{2k_n+1}}{X_{2k_n}} \right)^2 \geq 2\rho$$

for some $\rho > 0$ not depending on n . Hence $T_n(X_{2k_n+2}) \geq \rho$. Choose n so large that (16) holds and

$$|\phi_n(X_{2k_n+2})| < \rho/6, \quad |\phi_n(X_{2k_n+1})| < \rho/6.$$

Then

$$S_n(X_{2k_n+2}) \leq T_n(X_{2k_n+2}) + \rho/6, \quad S_n(X_{2k_n+1}) \geq T_n(X_{2k_n+1}) - \rho/6.$$

Taking into account that $2X_{2k_n+2} = X_{2k_n+1}$ so that

$$2S_n(X_{2k_n+2}) = S_n(X_{2k_n+1}),$$

it follows that

$$2T_n(X_{2k_n+2}) + \rho/2 \geq T_n(X_{2k_n+1})$$

and hence

$$\frac{T_n(X_{2k_n+1})}{T_n(X_{2k_n+2})} \leq \frac{5}{2}.$$

However, from $2X_{2k_n+2} = X_{2k_n+1}$ and the bounds on $A_n(x)$ and $B_n(x)$, we find that

$$\frac{T_n(X_{2k_n+1})}{T_n(X_{2k_n+2})} \geq 4 \left(\frac{8}{10} \right)^2,$$

which is a contradiction.

This proves the proposition. ■

5. PROOF OF THEOREM 1.5

By a straightforward change of coordinate systems Theorem 1.5 is equivalent to:

PROPOSITION 5.1. *Let (z_j) be a sequence in $\{|z| < 1\}$ with $z_j = \rho_j e^{i\phi_j} \neq 0$ ($j \geq 1$) such that $z_j \rightarrow 0$ ($j \rightarrow \infty$). Suppose that there is an $r > 0$ such that for every j_0 there exist $j, k \geq j_0$ such that*

$$|\sin(\phi_k - \phi_j)| \geq r.$$

Set $K = \{z_j\} \cup \{0\}$. Let $f(z)$, $z \in K$, be a function on K with $f(0) = 1$ and $f(z_j) \neq 0$ ($j = 1, 2, \dots$).

Let (p_n) be a sequence of polynomials whose zeros lie in $\{\operatorname{Im} z \geq 1\}$ and such that $p_n(z) \rightarrow f(z)$ ($n \rightarrow \infty$), pointwise on K . Then (p_n) converges locally uniformly in \mathbb{C} .

Proof of Proposition 5.1. We may assume that $p_n(0) = 1$ for $n \geq 1$ and then

$$p_n(z) = \prod_k (1 + w_{n,k} z).$$

If $w_{n,k} = w = u + iv$ corresponds to the zero $\xi + i\eta$, then

$$w = -\frac{1}{\xi + i\eta} = -\frac{\xi - i\eta}{\xi^2 + \eta^2}$$

and so

$$u = -\frac{\xi}{\xi^2 + \eta^2}, \quad v = \frac{\eta}{\xi^2 + \eta^2}.$$

Since $\eta \geq 1$ it follows that

$$0 < |w|^2 = \frac{1}{\xi^2 + \eta^2} \leq \frac{\eta}{\xi^2 + \eta^2} = v \leq \frac{1}{\eta} \leq 1. \tag{17}$$

Consider, for $|z| < 1$,

$$\phi_n(z) = \log p_n(z) = \sigma_{n,1}z + \sigma_{n,2}z^2 + \dots$$

with $\phi_n(z)$ the branch of the logarithm for which $\phi_n(0) = 0$ and

$$\sigma_{n,j} = \frac{(-1)^{j-1}}{j} \sum_k w_{n,k}^j.$$

From (17) we see that for $j \geq 2$,

$$|\sigma_{n,j}| \leq \frac{1}{j} \sum |w_{n,k}|^j \leq \frac{1}{j} \sum |w_{n,k}|^2 \leq \frac{1}{j} \sum v_{n,k} \leq \frac{1}{j} |\sigma_{n,1}|.$$

Hence, for $|z| \leq 1/2$,

$$\begin{aligned} |\sigma_{n,2}z^2 + \sigma_{n,3}z^3 + \dots| &\leq |\sigma_{n,1}| \frac{|z|^2}{2} (1 + |z| + |z|^2 + \dots) \\ &= |\sigma_{n,1}| \frac{|z|^2}{2(1 - |z|)} \\ &\leq |\sigma_{n,1}| |z|^2; \end{aligned}$$

and therefore for such z ,

$$\phi_n(z) = \sigma_{n,1}z + R_n(z), \quad |R_n(z)| \leq |\sigma_{n,1}| |z|^2. \tag{18}$$

Put $\sigma_{n,1} = M_n e^{i\theta_n}$. If (M_n) remains bounded we see from (18) that the functions $\phi_n(z)$ are uniformly bounded on $\{|z| \leq 1/2\}$, hence so are the polynomials $p_n(z) = e^{\phi_n(z)}$. Then as in the proof of Theorem 1.2 we conclude that (p_n) converges locally uniformly in \mathbb{C} .

We now assume that (M_n) is unbounded. Passing to a subsequence, we may assume that (θ_n) converges, say with limit θ . Let j_0 be such that for every $j > j_0$ we have $|z_j| < 1/2$ and $f(z_j) \neq 0$ and let $j > j_0$. From (18) it

follows that, for every n ,

$$\begin{aligned}\operatorname{Re}\{\phi_n(z_j)\} &= M_n \rho_j \cos(\phi_j + \theta_n) + \operatorname{Re}\{R_n(z_j)\}, \\ \operatorname{Re}\{R_n(z_j)\} &\leq M_n \rho_j^2,\end{aligned}$$

so that

$$\left| \frac{\operatorname{Re}\{\phi_n(z_j)\}}{M_n \rho_j} - \cos(\phi_j + \theta_n) \right| \leq \rho_j. \quad (19)$$

We have

$$\operatorname{Re}\{\phi_n(z_j)\} = \log|p_n(z_j)| \rightarrow \log|f(z_j)| \quad (n \rightarrow \infty);$$

so $(\operatorname{Re}\{\phi_n(z_j)\})_n$ remains bounded. Letting $n \rightarrow \infty$ in (19) it follows since (M_n) is unbounded,

$$|\cos(\phi_j + \theta)| \leq \rho_j,$$

where $\theta = \lim_n \theta_n$. Now let $j, k > j_0$ be such that $\sin(\phi_j - \phi_k) \geq r$, $\rho_j < r/4$, $\rho_k < r/4$, where r is the number in the proposition. Then

$$|\cos(\phi_j + \theta)| \leq r/4, \quad |\cos(\phi_k + \theta)| \leq r/4,$$

and

$$\begin{aligned}\cos(\phi_k + \theta) &= \cos(\phi_k - \phi_j + \phi_j + \theta) \\ &= \cos(\phi_k - \phi_j)\cos(\phi_j + \theta) - \sin(\phi_k - \phi_j)\sin(\phi_j + \theta)\end{aligned}$$

and so

$$|\sin(\phi_k - \phi_j)| |\sin(\phi_j + \theta)| \leq |\cos(\phi_k + \theta)| + |\cos(\phi_j + \theta)|.$$

Hence

$$r \left(1 - \frac{r^2}{16}\right)^{1/2} \leq \frac{r}{2},$$

which is impossible, since $0 < r < 1$. This contradiction implies that (M_n) is bounded and so the theorem is proved. ■

Remarks. (i) Let K be a set containing a complex sequence (z_j) with its limit $\zeta \in \{\operatorname{Im} z < 0\}$. Suppose $|z_j - \zeta|$ is decreasing. Combining Theorems 1.2 and 1.5 we see that if a sequence of polynomials with zeros in the closed upper half-plane exists which converges uniformly on K , but not

locally uniformly in \mathbf{C} , to a function f with $f(\zeta) \neq 0$, it is necessary that

- $\inf_j (|z_{j+1} - \zeta| / |z_j - \zeta|) = 0$,
- $\arg(z_j - \zeta)$ converges mod π .

In view of Proposition 4.1 and the Hermite–Biehler theorem, see [7, p. 256], these two conditions are not sufficient. It might be interesting to know necessary and sufficient conditions.

(ii) The construction given in the proof of Theorem 1.4 does not apply in case $\text{Im } \zeta < 0$. The following example gives a sequence (z_j) converging to ζ , $\text{Im } \zeta < 0$ satisfying both conditions above for which there exists a sequence of polynomials (p_n) with only real zeros which converges uniformly on K to 1 but not locally uniformly in \mathbf{C} .

For the example we use functions of the form e^{az} with $a \in \mathbf{R}$. Such a function belongs to $\mathcal{L}\text{-}\mathcal{P}$: it is the locally uniform limit of the polynomials $(1 + az/k)^k$. Hence it is clear that in the example the exponential functions can be replaced by such polynomials.

Let

$$\zeta = i, \quad z_j = i(2^{-\nu_j} - 1), \quad \nu_j \in \mathbf{N},$$

where (ν_j) is an increasing sequence of positive integers such that for a subsequence $j_1, j_2, \dots, \nu_{j_{n+1}} - \nu_{j_n} \rightarrow \infty$ ($n \rightarrow \infty$). Set

$$f_n(z) = \exp(2\pi \cdot 2^{\nu_{j_n}} z).$$

It is not difficult to show that $f_n(z) \rightarrow 1$ ($n \rightarrow \infty$) uniformly on $\{z_j\} \cup \{\zeta\}$, but (f_n) is not locally uniformly convergent in \mathbf{C} .

(iii) In Theorem 1.5 the condition $\text{Im } \zeta < 0$ plays a crucial role in the proof. The next example shows that this condition is also crucial for the conclusion of the theorem, i.e., it is no longer true in the case $\text{Im } \zeta = 0$.

Let (ν_j) be a sequence of positive integers with $\nu_{j+1} - \nu_j \rightarrow \infty$ ($j \rightarrow \infty$) and set

$$z_{2j-1} = \frac{1}{2^{\nu_j}} e^{i\pi/4}, \quad z_{2j} = \frac{1}{2^{\nu_j}} e^{i3\pi/4}.$$

Define

$$f_n(z) = e^{-2\pi 2^{2\nu_n} z^2}$$

and then $f_n(z) \rightarrow 1$ ($n \rightarrow \infty$) uniformly on $K = \{z_j\} \cup \{0\}$. However, (f_n) is certainly not locally uniformly convergent in \mathbf{C} . The functions f_n belong to $\mathcal{L}\text{-}\mathcal{P}$ so we may replace them by a suitable sequence of polynomials having only real zeros.

If we consider the sequence $(z_j + \zeta)$ converging to some ζ with $\text{Im } \zeta < 0$, then we have a sequence that satisfies the condition of Theorem 1.5. Hence it is clear that the cases $\text{Im } \zeta = 0$ and $\text{Im } \zeta < 0$ are significantly different.

(iv) Finally one might wonder if the requirement $f(\zeta) \neq 0$ of Theorem 1.2 is necessary and that the result might still be true if $f(z_j) \neq 0$ ($j \geq 1$), $f(z_j) \rightarrow 0$ ($j \rightarrow \infty$). The following example shows that the theorem is not true in this case. Let $z_j = i/2^j$ and $f_n(z) = z \cdot e^{2\pi 2^n z}$. Then $f_n(z) \rightarrow z$ ($n \rightarrow \infty$) uniformly on $K = \{z_j\} \cup \{0\}$ and $|z_{j+1}|/|z_j| = 1/2$, but (f_n) is not locally uniformly convergent in \mathbb{C} .

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REFERENCES

1. S. N. BERNSTEIN, Sur la distribution des zéros des polynomes tendant vers une fonction continue positive sur un segment donné, *J. Math. Pures Appl.* **8** (1929), 327–337.
2. J. KOREVAAR AND C. LOEWNER, Approximation on an arc by polynomials with restricted zeros, *Indag. Math.* **26** (1964), 121–128.
3. B. JA. LEVIN, On the convergence of sequences of H -polynomials, *Mat. Sb.* **66** (1965), 384–397. [In Russian]
4. E. LINDWART AND G. PÓLYA, Über einen Zusammenhang zwischen der Konvergenz von Polynomfolgen und der Verteilung ihrer Wurzeln, *Rend. Circ. Mat. Palermo* **37** (1914), 297–304.
5. P. MONTEL, "Leçons sur les Familles normales de Fonctions analytiques et leurs Applications," Gauthier-Villars, Paris, 1927.
6. G. PÓLYA, Über Annäherung durch Polynome mit lauter reellen Wurzeln, *Rend. Circ. Mat. Palermo* **36** (1913), 279–295.
7. G. PÓLYA AND G. SZEGÖ, "Aufgaben und Lehrsätze aus der Analysis, I," Springer-Verlag, Berlin, 1925.
8. H. L. ROYDEN, "Real Analysis," MacMillan, New York, 1970.
9. E. C. TITCHMARSH, "The Theory of Functions," Oxford Univ. Press, Oxford, 1964.